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AN APPROXIMATE THEORY FOR TURBULENT WAKES BEHIND HYPERSONIC BLUNT BODIES

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CONVAIR SAN DIEGO CONVAIR DIVISION GENERAL DYNAMICS CORPORATION

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ABSTRACT

The growth of the turbulent core in the constant pressure wake, of a blunt hypersonic body is predicted by means of a mixing length theory. Near the body where the core is smaller than the enthalpy wake the constant density theory of Townsend is applied. A variable density solution is obtained for the region in which the core and enthalpy wake are of the same size. The two solutions are fitted to predict velocities, temperatures, and concentrations in the constant pressure region for several thousand body diameters behind the missile.

I. INTRODUCTION

The environment around hypersonic bodies in the atmosphere is not well understood especially in the base and wake regions. In spite of the uncertainty associated with the chemical processes occurring in high temperature air, theoretical analyses of the shock layer and boundary layer have met with a great deal of success; whereas, analyses of the base and wake flow have not been particularly rewarding. This is unfortunate when attempting to detect and discriminate between bodies entering the earth's atmosphere because the wake comprises the largest portion of the observable flow field. In the present analysis, a simple model of the turbulent wake has been used to calculate the properties in the wake. The effect of a few foreign materials in the wake is discussed because they may have

a profound effect on the electromagnetic, radiative, and transport properties of the flow. A foreign material (or additives as they are referred to) is any chemical substance not normally found in the atmosphere. An understanding of the effect of additives on the environment around hypersonic vehicles may be necessary to solve the problems of detecting, discriminating, and destroying re-entering ballistic missiles.

Feldman[1] and Goulard[2] have obtained solutions for the physical properties in a laminar trail behind a missile. Laminar wakes are very long; the length being proportional to the frontal area of the missile. Feldman's calculations show that at an altitude of 60,000 ft. and a velocity of 17,500 ft/sec the temperature 1,000 km behind a 1-m-diam missile is 1500°C, or about one-half the wake temperature near the missile. Laminar solutions also indicate that the width of a hypersonic wake is nearly constant. Townsend[3] developed a mixing length theory which predicts velocities in the turbulent wake behind a body passing through an incompressible fluid. In this case the length of the trail is proportional to the diameter of the missile, not its area, and the width of the turbulent zone increases as the cube root of distance behind the body. Townsend's incompressible theory was modified[4,5] and applied to hypersonic wakes. The present work is based on this modification.

The problems of transition from laminar to turbulent flow in the flow field are not considered. It is assumed the wake is turbulent over its entire length. The wake is not, however, turbulent over its entire width at all points.

II. FLOW FIELD

Before presenting the theory of turbulent diffusion in a hypersonic wake the properties of the wake will be discussed qualitatively. The general properties of a hypersonic wake are outlined in Figure 1. The flow behind a missile can be divided in the axial direction into three regions; the base region, the expansion

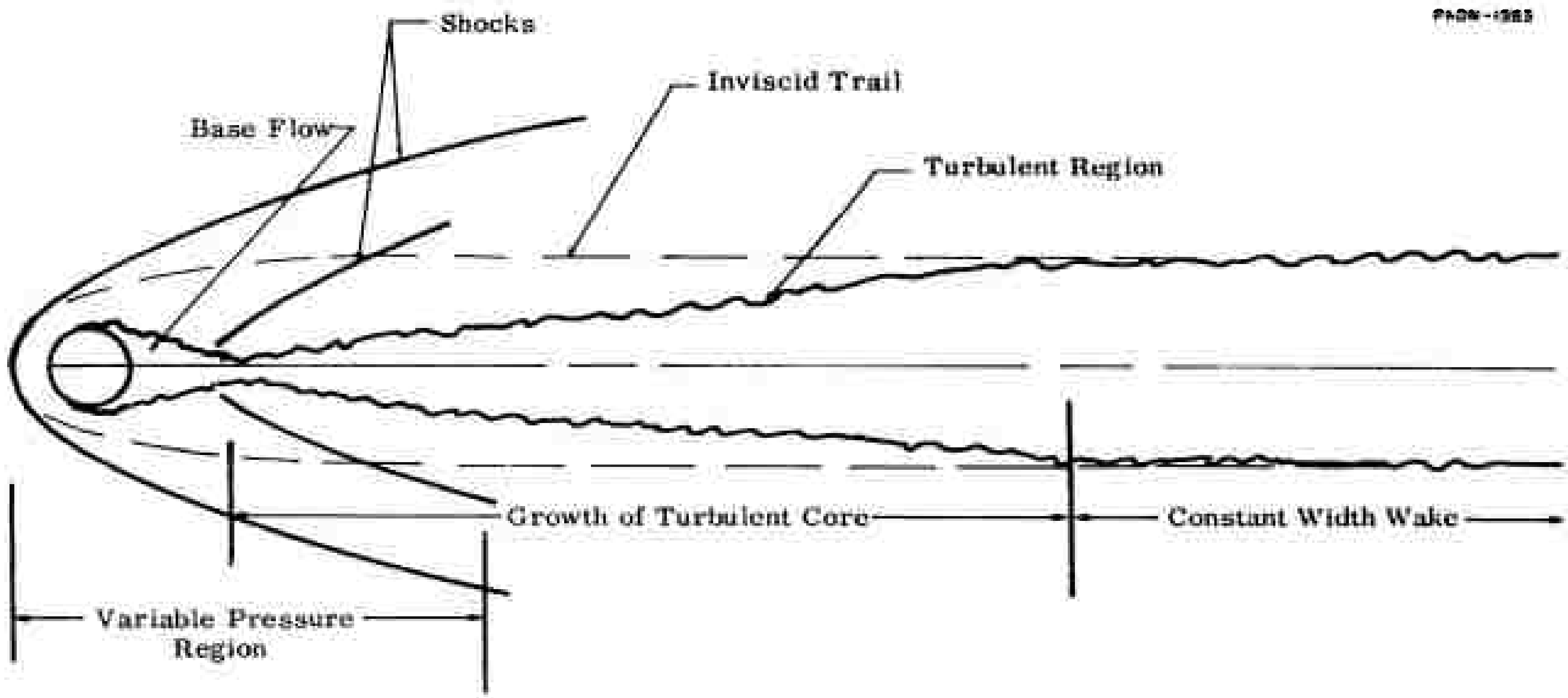

FIGURE 1 TURBULENT WAKE BEHIND A HYPERSONIC BODY

wake, and the constant pressure wake. The constant pressure region is often termed the far wake. The wake can be further divided in the radial direction into the viscous and inviscid flow. The turbulent core constitutes the region in which viscous forces are important. Surrounding the core is the inviscid region which is caused mainly by the bow shock. It is the growth of the turbulent zone into the inviscid flow and the consequent diffusion of the high enthalpy trail which will be discussed here.

The inviscid flow behind a blunt body can be characterized by a maximum velocity, U_o, which is about one-fifth of the missile velocity, a maximum temperature, T_o, which is about one-half of the stagnation temperature, and a width, L_o, which is several times that of the body. Figure 2 shows a cross section of the viscous and inviscid portions of the wake behind a spherical body having a velocity of 17,500 ft/sec. The large variations in velocity, temperature, and concentration near the axis originate in the boundary layer at the surface of the missile. The boundary layer on the body will flow into the turbulent core at the axis of the wake. Temperatures in the turbulent core are sensitive to the type of surface material on the body. If the body surface were adiabatic the gas temperature in the base region and at the axis of the core immediately downstream of the base would be very close to the stagnation temperature. In practical cases, the surface material absorbs energy from the boundary layer, and the gases radiate energy to the surroundings; therefore, the temperature at the axis of the turbulent core may be closer to that of the missile surface than the stagnation temperature. All material which flows into the boundary layer on the missile will be deposited in the turbulent core. Therefore, material ablated from the missile surface will be deposited in the core as shown by the concentration curve in Figure 2. Thus, only the properties of the boundary layer and core of the wake will be affected by ablation. However, some properties, the electron concentration for example, may be very sensitive to small amounts of foreign materials present. The dimensions and thermodynamic properties of the core are, therefore, of major interest.

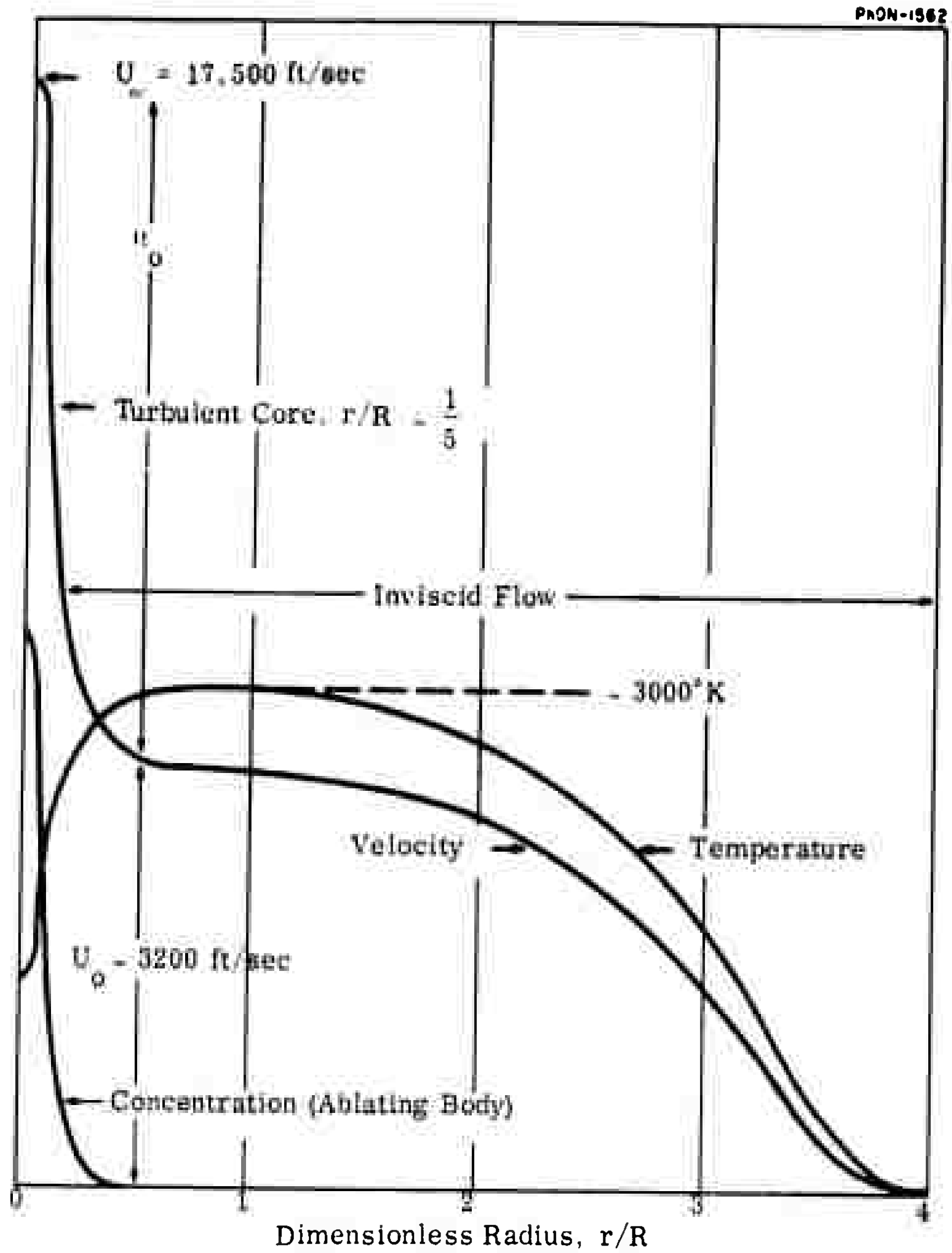

FIGURE 2 VELOCITY, TEMPERATURE, AND CONCENTRATION BEHIND A HYPERSONIC BLUNT BODY

III. ASSUMPTIONS

It is assumed in this analysis that the core of the wake is completely turbulent. This assumption is based on pictures obtained in hypersonic ranges at 1 atm. The possibility of transitions from laminar to turbulent flow occurring in the wake behind a missile at high altitudes is neglected. Pressure variations in the expansion wake are not included in the analysis. The results of radiation measurements made by Georgiev[6] in a hypersonic range indicate that the point at which the pressure is nearly equal to ambient pressure is approximately 40 diam behind the missile. Errors contributed by the assumption of constant pressure in the first 40 diam will not greatly affect the dimensions of the total wake which is several thousand diameters in length. The increase in temperature by viscous dissipation in wakes was found by Feldman[1] to be small and is neglected. Temperatures in a wake are in general less than 3000°K for re-entry speeds; therefore, the wake was treated as a perfect gas. The thermodynamic properties of high temperature air,[7] however, were used to obtain the initial properties of the wake.

The exact nature of the turbulent core will depend upon the geometry and the flight conditions of the body. Because of the wide variety of temperature profiles possible in the turbulent core, it has been assumed to be initially isothermal in this calculation. Therefore, the turbulent core analysed here will consist of a sharp velocity profile as depicted in Figure 2 in a uniform temperature and pressure field. The early growth of the turbulent core, in this case, can be estimated by the incompressible theory developed by Townsend.[3] It is assumed that this analysis is valid to the point at which the width of the turbulent core is equal to that of the inviscid wake; beyond this point a different analytical solution will be obtained.

The assumptions used are summarized by the following list:

1. The fluid is a perfect gas;

2. The wake is axi-symmetric;

3. The core is completely turbulent;

4. Constant pressure,

5. Negligible radial velocity components;

6. Steady state;

7. Negligible viscous dissipation;

8. Prandtl mixing length theory;

9. Eddy Prandtl number is unity, $\frac{C_p \epsilon_\mu}{\epsilon_k} = 1$

IV. ANALYSIS

Townsend has shown by means of a mixing length theory that when self-preserving flow is established the velocity distribution is given by

$$u = u_a \exp \left[-\frac{1}{2} \frac{r^2}{\ell^2} \right] \tag{1}$$

The nomenclature is listed on p. 17. The lower case u represents the velocities in the core relative to the maximum velocity of the inviscid flow at the same point, U, assuming the core did not exist. The total velocity at any point is, therefore, u + U. The width of the core, ℓ, and the velocity at the axis, u_a, are given by

$$\ell^3 = \frac{3\, F\, (x - x_o)}{2\, \pi \rho\, U_\infty^2\, R_t} \tag{2}$$

and

$$u_a^3 = \frac{F\, U_\infty\, R_t^2}{\rho\, 18 \pi\, (x - x_o)^2} \tag{3}$$

Equations (2) and (3) include the drag, F, which in the hypersonic case is the drag experienced by the core which is the skin friction and base drag but not the wave

drag. Therefore, F can be expressed in terms of the initial width and velocity of the core by using the following momentum integral

$$F = 2\pi\rho\, U_o^2 \int_o^\infty \frac{u}{u_o}\left(1 - \frac{u}{u_o}\right) r\,dr \tag{4}$$

A solution to this integral equation can be obtained by assuming a Gaussian velocity distribution as used by Townsend in Equation (1). The drag is

$$F = \pi\rho\, u_o^2\, \ell_o^2 \tag{5}$$

The core width and maximum velocity can now be expressed by

$$\ell^3 = \frac{3\,\ell_o^2 (x - x_o)}{2\,R_t} \tag{6}$$

and

$$u_a^3 = \frac{R_t^2\, u_o^3\, \ell_o^2}{18\,(x - x_o)^2} \tag{7}$$

Equations (1), (6), and (7) provide an approximate method for predicting velocities in a turbulent wake to the point where the width of the turbulent core has grown to equal that of the inviscid wake. The turbulent core has nearly disappeared from the velocity profile because the momentum of the core is a small fraction of the total wake momentum.

A solution will now be obtained for the decay of the fully turbulent compressible wake. This solution and that given above can be joined to give a complete description of the wake. By using the assumption listed earlier the momentum equation can be simplified to give

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial r} = \frac{1}{r\rho}\,\frac{\partial}{\partial r}\,(r\tau), \tag{8}$$

and the energy equation is

$$U \frac{\partial h}{\partial x} + V \frac{\partial h}{\partial r} = \frac{1}{r\rho} \frac{\partial}{\partial r} (r q) \tag{9}$$

The distances x and r can be considered dimensionless since each term contains x^{-1} or r^{-1}. When using the concept of eddy viscosity and eddy conductivity the shear and energy flux are defined by

$$\tau = \rho \epsilon_\mu \frac{\partial U}{\partial r} \tag{10}$$

and

$$q = \rho \epsilon_k \frac{\partial T}{\partial r} \tag{11}$$

Following the mixing length theory used by Townsend it is assumed that the eddy viscosity is proportional to the wake width and the greatest velocity difference across the wake.

The following relation will be used

$$\epsilon_\mu = \frac{L U_a}{R_t} \tag{12}$$

Where R_t is the same unknown constant obtained for the incompressible case. Using this relation, the fact that $h = C_p T$ for a perfect gas, and assumption 5 Equations (8) and (9) can be simplified to give

$$U_\infty \frac{\partial U}{\partial x} = \frac{\epsilon_\mu}{r\rho} \frac{\partial}{\partial r}\left(r\rho \frac{\partial U}{\partial r}\right) \tag{13}$$

and

$$U_\infty \frac{\partial T}{\partial x} = \frac{\epsilon_k}{C_p r\rho} \frac{\partial}{\partial r}\left(r\rho \frac{\partial T}{\partial r}\right) \tag{14}$$

The velocity term on the left-hand sides of the momentum and energy equations should be U; however, it was assumed to be U_∞ because the maximum velocity at the axis of the fully turbulent wake is within 20% of the free stream velocity.

The solution of Equations (13) and (14) will be similar provided the eddy Prandtl number is unity and the initial conditions are similar. If these two conditions are true the temperatures and velocities in the wake can be related in the following manner:

$$U = a\,(T - T_\infty) \tag{15}$$

Substitution of Equation (15) into Equation (12) gives

$$\epsilon_\mu = \frac{a\,L\,(T_a - T_\infty)}{R_t} \tag{16}$$

Combining Equation (16) and the perfect gas law with Equation (14) produces the following differential equation in terms of temperature

$$\frac{R_t\,U_\infty}{a\,L\,(T_a - T_\infty)}\,\frac{\partial T}{\partial x} = \frac{T}{r}\,\frac{\partial}{\partial r}\left(\frac{r}{T}\,\frac{\partial T}{\partial r}\right) \tag{17}$$

This equation can be solved by separation of variables for the case when $T_\infty = 0$. This case is of interest because the ambient temperature is very low ($T_\infty \sim 200\,°K$) compared with temperatures in the wake of re-entering ballistic missiles ($T_o \sim 3000\,°K$). Assuming that $T = f_1(x)\,f_2(r)$ gives the following relations:

$$f_1 = e^{-\frac{1}{\alpha^2}\int \frac{dx}{A(x)}} \tag{18}$$

and

$$f_2 = b\,e^{-\left(\frac{r}{2\alpha}\right)^2} \tag{19}$$

It is believed that the mixing length is directly related to the width of the wake. By defining the mixing length in compressible flow, L, as

$$L = \int_0^\infty \frac{T}{T_a}\,dr \tag{20}$$

the variable $A(x)$ can be evaluated to be

$$A(x) = \frac{1}{\alpha}(x + c) \tag{21}$$

where c is a constant. Substitution of this relation into Equation (18) gives an equation for f_1 in terms of x. The temperature is approximated by

$$T = \frac{\alpha R_t U_\infty}{a\sqrt{\pi}\,(x + c)} \; e^{-\left(\frac{r}{2\alpha}\right)^2} + T_\infty \tag{22}$$

when $T_\infty << T_0$. Equation (22) only satisfies the original differential equation [Equation (17)] exactly when $T_\infty = 0$. According to Equation (16) when $T_a \rightarrow T_\infty$, $\epsilon_\mu \rightarrow 0$; therefore, the concept of eddy properties becomes invalid. The eddy viscosity can be greater than the molecular viscosity, ν, but it cannot be less. When $\epsilon_\mu \rightarrow \nu$ the momentum and energy equations for laminar flow must be employed. At distances greater than 1000 diameters the constant width wake predicted by Equation (22) has cooled and begins to grow again; therefore, this analysis cannot be extended beyond this point.

Wake temperature obtained by means of Equations (1) and (22) will always follow a Gaussian distribution. This would require that the initial condition have the same form. The radial temperature distribution at x = 0, the point where the pressure equals the ambient pressure, has been computed by assuming an elliptical shaped shock wave around the nose of the missile as suggested by Van Dyke.[8] If it is assumed that the only irreversible processes occurs at the shock wave, the thermodynamic state of the gas at x = 0 can be computed by following an isentropic expansion to atmospheric pressure along each streamline. The velocity and position of each streamline which passes through the shock wave can be located at x = 0 by employing the Bernoulli and continuity equations. The result of such a calculation is shown as curve x = 0 in Figure 3, which gives the properties of the inviscid flow. The

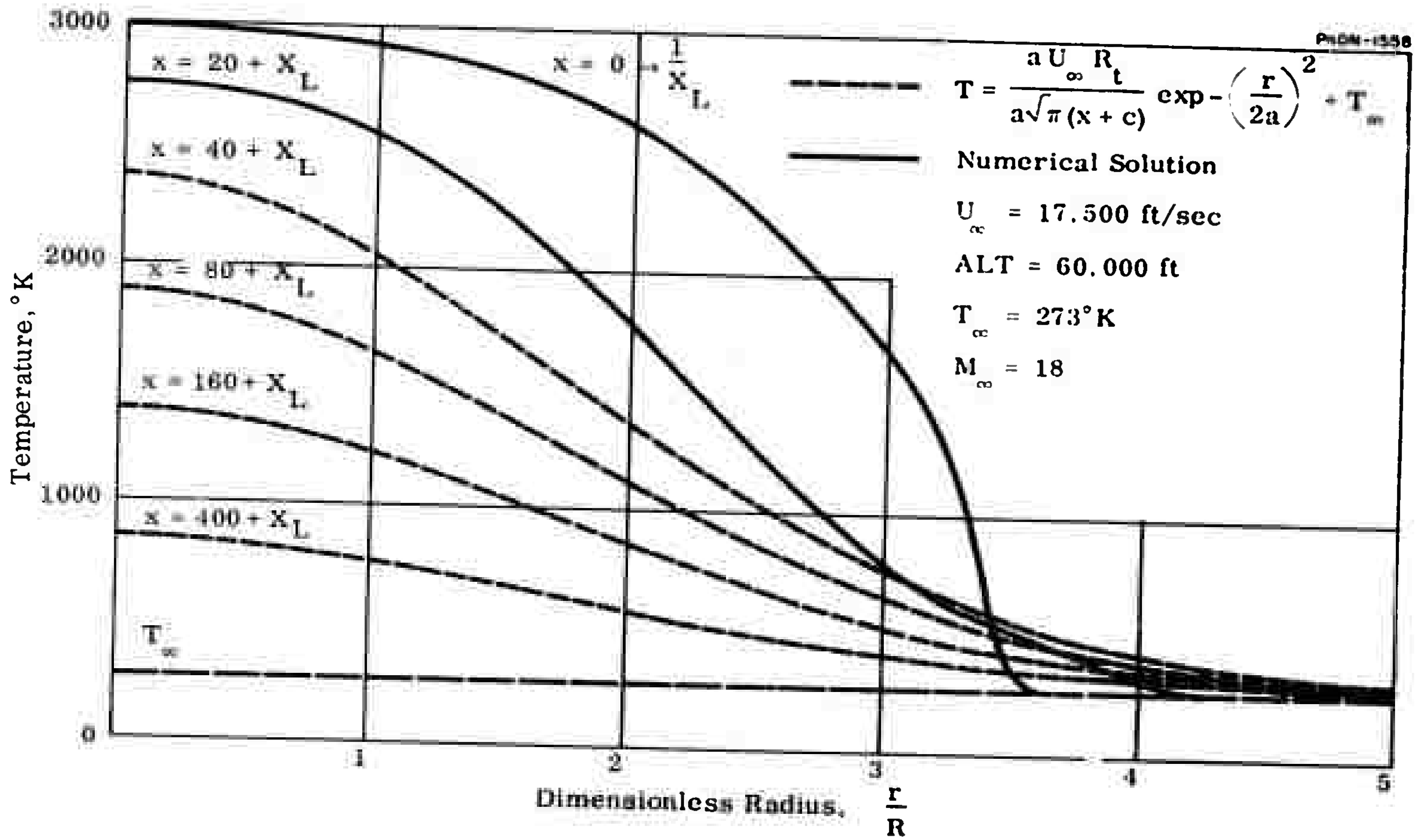

FIGURE 3 TEMPERATURES IN A TURBULENT WAKE

properties of the turbulent core were obtained by assuming $u_o = U_\infty - U_o$ and its momentum equal that of the boundary layer on the body at its point of separation.

The constants α and c in Equation (22) were evaluated by fitting the initial temperature profile. It is obvious in Figure 3 that the curve labeled x = 0 does not follow a Gaussian. A finite difference solution to Equation (17) using the conditions at x = 0 was carried out using a digital computer to determine how quickly this curve approached a Gaussian distribution. The results of this numerical solution showed the initial conditions decay to a curve which can be fitted by Equation (22) within 20 diam. This decay does not begin until the core has grown to the width of the inviscid wake, L. A constant, X_L, which is the length of the wake required for the core to grow to width L, is added to the dimensions shown in Figure 3.

Townsend computed theoretically that the constant R_t has a value between 14 and 21. A value of 14 was used in the present calculations. Results obtained by means of Equations (1), (6), (7), and (22) are shown in Figures 4 and 5 for a missile velocity of 17,500 ft/sec, an altitude of 60,000 ft, and an initial core diameter of 1/5 the missile diameter.

V. CONCLUSIONS

The present analysis is approximate and subject to all the short comings of a mixing length theory. Several sources of uncertainty are outlined here.

The turbulent diffusivity is assumed to be a function of the wake width and maximum velocity difference, which is the basis of most mixing length theories. This implies that the turbulent properties depend on the axial distance, are constant in the radial direction, and are in local equilibrium. One might expect the turbulence to lag behind the equilibrium value where the wake is growing rapidly.

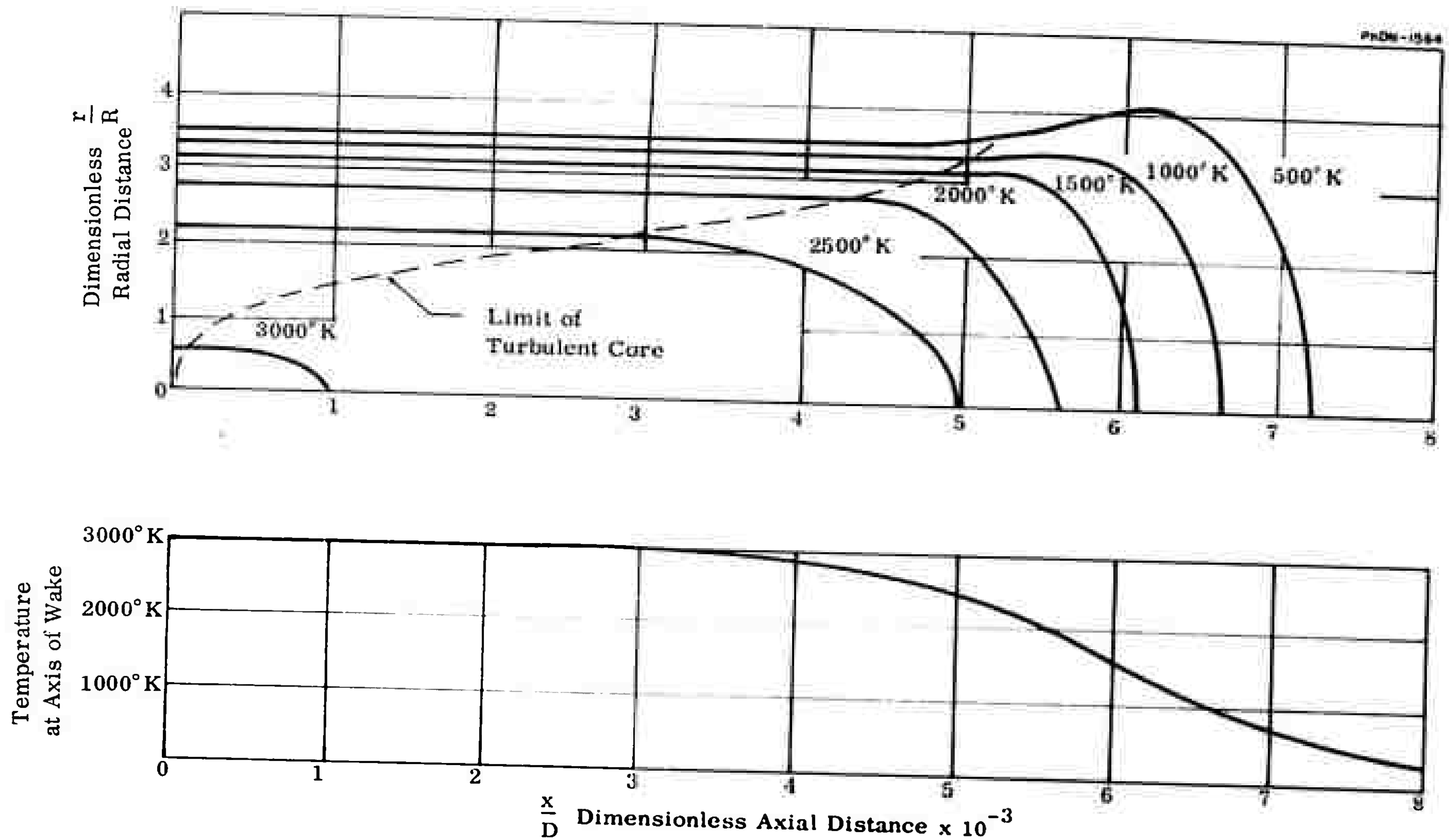

FIGURE 4 APPROXIMATE TEMPERATURES IN WAKE OF MISSILE

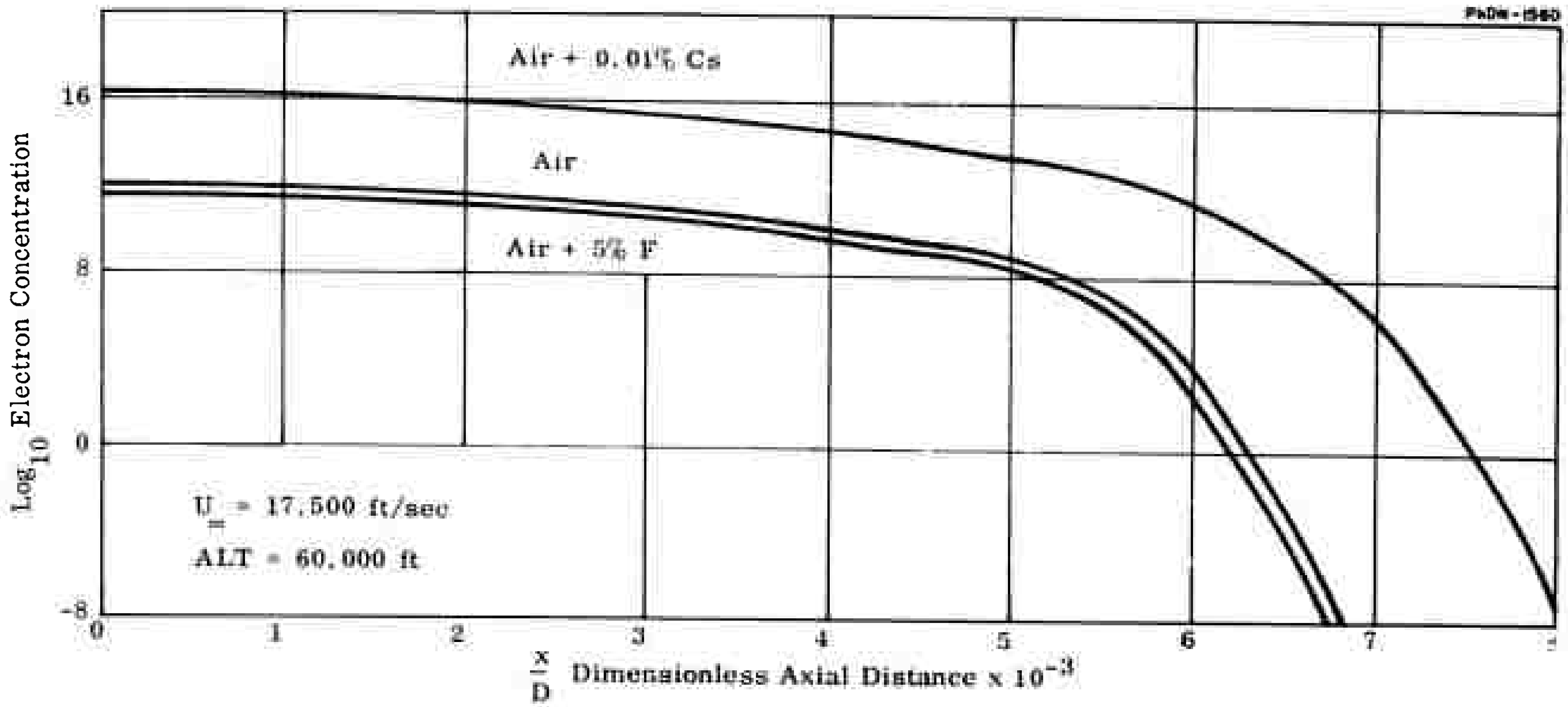

FIGURE 5 ELECTRON WAKE BEHIND A RE-ENTRY BODY

The assumption of an incompressible core becomes less valid as the point at which the core and inviscid wakes are of equal width is approached. As the turbulent core reaches the edges of the wake the velocity difference in the turbulent region increases; therefore, the turbulence level might be expected to increase in this region of increased shear and cause the core to grow faster than predicted.

The wakes computed here, while only approximate, demonstrate the turbulent wake is much shorter than the laminar wake. The physical model and the value of R_t can be evaluated when ballistic range data are available.

Equation (22) predicts a constant width wake at large distances from the projectile because it was assumed that $T_\infty << T_a$. Under this condition it is true the hot turbulent wake will grow very slowly; however, when $T_\infty \approx T_a$ the wake begins to grow again as the cube root of axial distance. These approximations, except the assumption of local equilibrium, tend to produce wakes which grow slower and, therefore, are longer than a more rigorous mixing length theory would predict. Assuming the turbulence to be in local equilibrium would give rise to shorter wakes than a lagging turbulence level.

NOMENCLATURE

a, b, c, x_o	constants
D	body diameter
C_p	specific heat at constant pressure
F	drag
h	enthalpy
ℓ	width of core
L	width of inviscid wake
q	heat flux
r	radial distance
R	body radius
R_t	turbulent Reynolds number = $\frac{U_a \ell}{\epsilon_\mu}$
T	temperature
u	axial velocity in core
U	axial velocity in inviscid flow relative to atmosphere
V	radial velocity in inviscid flow
x	axial distance
X_L	distance at which $\ell = L$ behind the body
ϵ_μ	turbulent viscosity
ϵ_k	turbulent conductivity
ρ	density
τ	shear

Subscripts

a	value at $r = o$
o	value at $x = o$, $v = o$
∞	free stream value

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